# Signal detection via residence times statistics: Noise-mediated minimization of the measurement error

A. Nikitin,[1] N. G. Stocks,[1] and A. R. Bulsara[2]
[1]*School of Engineering, University of Warwick, Coventry CV4 7AL, United Kingdom*
[2]*Space and Naval Warfare Systems Center 2363, 49590 Lassing Road, RM 341, San Diego, California 92152-6147, USA*
(Received 16 December 2002; published 26 September 2003)

We study the problem of detecting a small dc signal by quantifying its effect on the mean difference $\Delta T$ in residence times in the stable steady states of a bistable dynamical measurement device, in the presence of a noise floor and a known time-sinusoidal bias signal. Errors in the measurement process occur due to a finite observation time that is present in most practical scenarios. The error is found to have a nonmonotonic dependence on the noise intensity; at a critical noise intensity, the error is minimized. This phenomenon, reminiscent of the well-known *stochastic resonance* effect, can also be obtained by adjusting the device tuning parameters for a given noise floor. The effect appears to be most pronounced for subthreshold bias signals in the strongly nonlinear response regime.

DOI: 10.1103/PhysRevE.68.036133 PACS number(s): 02.50.Ey, 05.40.−a, 06.20.Dk

A constant external signal can break the symmetry of nonlinear systems; this phenomenon manifests itself, e.g., as a nonzero current in stochastic ratchets [1], the appearance of even harmonics of a known time-sinusoidal bias signal in the power spectral density (PSD) [2], or a nonzero difference in the average residence times in the stable attractors of a two-state system [3,4]. The appearance of these features can be used as a means of detecting and quantifying a "target" dc signal [3–5] (e.g., conventional fluxgate sensors use the appearance of even harmonics in the PSD [6]). However, in most applications, one can only observe the signal for a finite time interval $T_{ob}$, thereby raising the spectre of a statistical error in the detection procedure. This error depends on the length of the time series (i.e., directly on $T_{ob}$) as well as the system and the noise parameters. It seems reasonable, therefore, to assume that the error could be minimized by a suitable adjustment of these parameters.

Here, we consider a bistable dynamical detector. In the absence of an asymmetrizing dc signal, the mean residence times (and their density functions) in the stable steady states are identical. The presence of the dc signal leads to a finite difference in the mean residence times together with a splitting of the associated density function. These phenomena can be quantified using a simple counting circuit, which allows one to estimate the target signal from the mean residence times difference $\Delta T$. The procedure has been successfully utilized in a real dynamical device, a fluxgate magnetometer [4], and boasts several advantages over conventional (PSD-based) detection schemes. We will show that the measurement error, due to a finite $T_{ob}$, exhibits a nonmonotonic dependence on the noise intensity, i.e., *there is an optimal noise intensity for which the error is minimized*.

Our prototype system consists of the standard [7] overdamped Duffing oscillator underpinned by the bistable potential energy function $V(x,t)=-(a/2)x^2+(b/4)x^4+cx-Ax\cos(\Omega t)$, that includes a *known* external bias signal $A\cos(\Omega t)$ in a zero-mean white noise floor $\xi(t)$ with intensity $2D$:

$$\dot{x}=-\frac{\partial V(x,t)}{\partial x}+\xi(t). \tag{1}$$

The system asymmetry enters via the small (compared to the energy barrier height in the absence of the periodic bias) target signal $c$; for $c\neq 0$ the potential $V(x,t)$ is skewed with the potential wells having unequal depths in the absence of the periodic bias. We set $a=b=1$ throughout this work.

In Ref. [8] we computed the residence times distributions as well as the mean residence times difference $\Delta T$ in the adiabatic regime and the weak noise but strong forcing limit (i.e. for $A/D\gg 1$ with $A$ subthreshold.) In this regime, one appears to get the maximal (nonlinear) shift $\Delta T$, for a given target signal $c$; accordingly, this is the regime for which we compute the measurement error stemming from the finite observation time. We note that the notion of the (finite) observation time was introduced in Ref. [4], in connection with the definition of a response signal-to-noise ratio for the case of a strongly suprathreshold periodic bias signal; in this regime—also the regime of our experiments [4]—$\Delta T$ is proportional to $c$. Before studying the measurement error (in $\Delta T$), we outline the main results from our earlier work [8].

The residence time distributions (RTDs) in the metastable states are well approximated by sums of Gaussians [8]

$$P_1(\tau)=(1-p)\sum_{m=0}^{\infty}\frac{p^m}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{[\tau-(m+1/2)T+\delta]^2}{2\sigma^2}\right) \tag{2}$$

for state 1; the RTD of state 2, $p_2(\tau)$, is identical but with the substitutions $p\rightarrow q$, $\delta\rightarrow-\delta$. The variables $1-p$ and $1-q$ designate the probabilities to escape states 1 and 2, respectively, in the first period $T$; these probabilities correspond to the area under the first peak of the RTD (shown by shaded regions in Fig. 1); $\sigma$ is the width (standard deviation) of the peaks; $\delta$ is the asymmetry-induced shift in the peaks from locations $(m+1/2)T$. It is straightforward to calculate [8] the first moments of the RTDs, $\langle\tau_1\rangle$ and $\langle\tau_2\rangle$, from Eq. (2): $\langle\tau_1\rangle=\alpha_p T/2+\delta$, $\langle\tau_2\rangle=\alpha_q T/2-\delta$, where $\alpha_{p,q}=(1+p,q)/(1-p,q)$. Hence, the average difference in residence times is

 

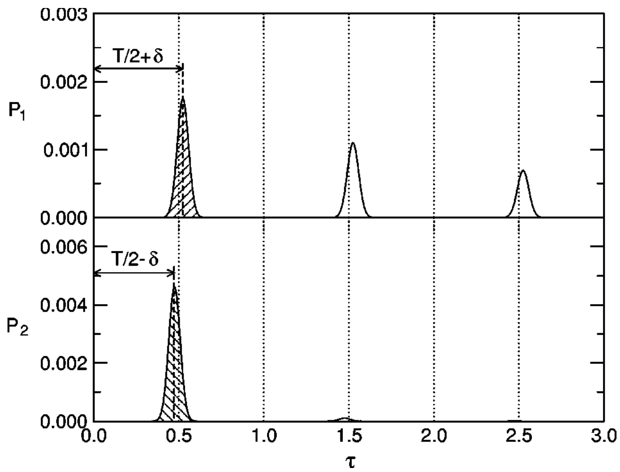


FIG. 1. The residence times probability densities $P_1$ and $P_2$ obtained from Eq. (2). The parameters are $a=1$, $b=1$, $c=0.012$, $A=0.35$, $\Omega=0.0025$, and $D=0.003$.

$$\Delta T=(\alpha_p-\alpha_q)T/2+2\delta. \tag{3}$$

The detection/quantification of the constant signal $c$ hinges on the functional interrelation between it and the nonzero difference $\Delta T$ in mean residence times. We now consider the normalized standard error, that results from the finite observation time $T_{ob}$, when making a measurement of $\Delta T$.

Let $\tau_{i,n}$ denote the measured residence times (for states $i\in\{1,2\}$) from which the average residence times are computed. Then for $N$ residence time measurements for each state, we have

$$T_{ob}=\sum_{n=1}^{N}\tau_{1,n}+\sum_{n=1}^{N}\tau_{2,n}. \tag{4}$$

The average values of the residence times, which include $N$ elements, are $\overline{\tau_i}=(1/N)\Sigma_{n=1}^{N}\tau_{i,n}$ with the differences between the average values (recall that $\langle\cdots\rangle$ denotes the theoretical mean value computed directly from the RTD) $\overline{\tau_i}-\langle\tau_i\rangle=\delta_{\tau_i}$, yielding the errors in the measurement of each mean residence time. Assuming the independence of the residence times, and using the equations $\langle\tau_{i,n}\rangle=\langle\tau_i\rangle$ we obtain for the average errors and their variances

$$\langle\delta_{\tau_i}\rangle=\langle\overline{\tau_i}\rangle-\langle\tau_i\rangle=\frac{1}{N}N\langle\tau_i\rangle-\langle\tau_i\rangle=0, \tag{5}$$

$$\begin{aligned}\sigma_i^2&=\langle(\delta_{\tau_i}-\langle\delta_{\tau_i}\rangle)^2\rangle=\langle(\overline{\tau_i}-\langle\tau_i\rangle)^2\rangle\\&=\frac{1}{N^2}\sum_{n=1}^{N}\int_0^\infty(\tau_{i,n}-\langle\tau_i\rangle)^2P_i(\tau_{i,n})d\tau_{i,n}\\&=\frac{1}{N}\left(\sigma^2+T^2\frac{\tilde{p}}{(1-\tilde{p})^2}\right),\end{aligned} \tag{6}$$

where we have used the shorthand notation $\tilde{p}\equiv(p,q)$ for $i=(1,2)$, respectively. We now introduce the average difference in the residence times in the metastable states; these consist of $N$ residence times in each state:

$$\Delta\overline{T}=\frac{1}{N}\sum_{n=1}^{N}(\tau_{1,n}-\tau_{2,n})=\frac{1}{N}\sum_{n=1}^{N}\Delta T_n.$$

Assuming $\langle\Delta T_n\rangle=\Delta T$ we obtain the dispersion

$$\sigma_{\Delta T}^2=\langle(\Delta\overline{T}-\Delta T)^2\rangle=\sigma_1^2+\sigma_2^2=\frac{1}{N}\left(2\sigma^2+\frac{T^2p}{(1-p)^2}+\frac{T^2q}{(1-q)^2}\right). \tag{7}$$

We also note that observation time $T_{ob}$ is a random variable [see the definition (4)] with average value

$$\begin{aligned}\langle T_{ob}\rangle&=\sum_{n=1}^{N}\langle\tau_{1,n}\rangle+\sum_{n=1}^{N}\langle\tau_{2,n}\rangle=N(\langle\tau_1\rangle+\langle\tau_2\rangle)\\&=N(\alpha_p+\alpha_q)\frac{T}{2}.\end{aligned} \tag{8}$$

Then, using Eqs. (3), (7), and (8) we obtain the dimensionless standard deviation (or the error of the measurement):

$$E_r=\frac{\sigma_{\Delta T}}{\Delta T}=\frac{1}{(\alpha_p-\alpha_q)T/2+2\delta}\times\sqrt{\frac{(\alpha_p+\alpha_q)T}{2\langle T_{ob}\rangle}\left(2\sigma^2+\frac{pT^2}{(1-p)^2}+\frac{qT^2}{(1-q)^2}\right)}. \tag{9}$$

It is clear that $E_r\propto\langle T_{ob}\rangle^{-1/2}$ and therefore $\lim_{\langle T_{ob}\rangle\to\infty}E_r=0$, i.e., the error decreases with increasing observation time, as should be expected. As a final note, we point out that expression (9) is obtained for fixed $N$. Generalizing $N$ to mean the average number of switches between states in the fixed observation time $T_{ob}$, we may rewrite $\langle T_{ob}\rangle$ as $T_{ob}$ in Eq. (9), valid when $T_{ob}\gg(\langle\tau_1\rangle+\langle\tau_2\rangle)$ (however, it should be noted that expression (9) is valid for arbitrary $T_{ob}$ and hence $N$).

Figure 2 shows the main result. We plot the error as a function of the noise intensity parameter $D$; good agreement between the theory and simulations is observed. The numerical results are obtained via the following technique: for every chosen noise intensity $D$, the numerical experiments are done $M$ times, and the averaged quantities $T_{obm}=(1/N)\Sigma_{n=1}^{N}(\tau_{1,n}+\tau_{2,n})_m$ and $\Delta\overline{T}_m=(1/N)\Sigma_{n=1}^{N}(\tau_{1,n}-\tau_{2,n})_m$ $(m=1,2,3,\ldots,M)$ are calculated. In terms of these averages we obtain $\langle T_{ob}\rangle_M=(1/M)\Sigma_{m=1}^{M}T_{obm}$, $\Delta T_M=(1/M)\Sigma_{m=1}^{M}\Delta\overline{T}_m$, and $\sigma_{\Delta T_M}^2=(1/M)\Sigma_{m=1}^{M}(\Delta\overline{T}_m-\Delta T_M)^2$. The final expression for the numerically computed error is then,

$$E_r(T_{ob}/T)^{1/2}\simeq\frac{\sigma_{\Delta T_M}}{\Delta T_M}\sqrt{\frac{\langle T_{ob}\rangle_M}{T}}. \tag{10}$$

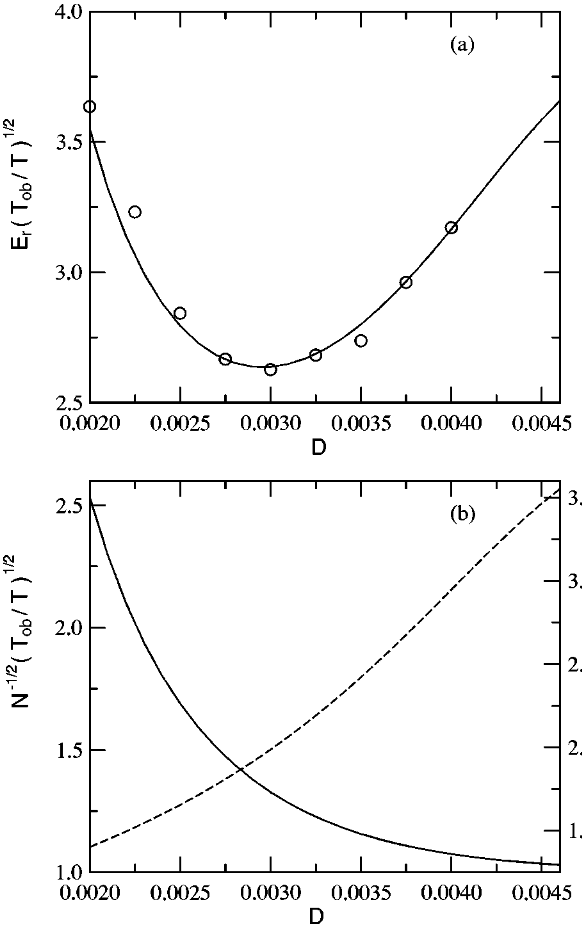


FIG. 2. (a) The dimensionless standard deviation (or the error) $E_r$ vs noise intensity parameter $D$. (b) The error $E_{r_1}$ (dashed line) and $1/\sqrt{N}$ (solid line) in the same (normalized) units. The crossover point approximately corresponds to the location of the minimum in (a). $A=0.35$, $\Omega=0.0025$, and $c=0.005$. Theoretical results were obtained from Eq. (9) (using expression for $p,q,\delta,\sigma$ obtained in Ref. [8]), data points are from digital simulation.

In all the figures we ensure that, given the asymmetry-inducing signal $c$, the bias signal amplitude $A$ is selected so that the Kramers approximation [9] to the hopping rate always remains valid: $A$ can never be large enough to cause deterministic switching, and the noise intensity should not exceed the (remaining) energy barrier height. One readily shows that the potential well at $x_i (i=1,2)$, vanishes at $A_{ci} = |c-(-1)^i (2/3)a\sqrt{a/3b}|$, so that we always maintain $A < \min(A_{c1},A_{c2})$.

Figure 2(b) shows the origin of the minimum error which is observed in Fig. 2(a). It is easy to see that the error $E_r$ can be presented as $1/\sqrt{N}$ times the error $E_{r_1}$ for $N=1$:

$$E_r = \frac{1}{\sqrt{N}} E_{r_1}. \tag{11}$$

For small $D$, transitions between the states are rare and hence $1/\sqrt{N}$ is large. As $D$ is increased, $1/\sqrt{N}$ rapidly decreases (approximately exponentially) until the system begins to synchronize to the periodic bias signal—this causes $N$ to change relatively slowly over a range of $D$. In fact, the number of transitions in the time $T_{ob}$ becomes independent of $D$ (for not too large $D$), saturating at $N/(T_{ob}/2)$. However, in this region, $E_{r_1}$ increases due to an increase in $\sigma$, hence, the normalized error $E_{r_1}$ increases [Fig. 2(b)]. Hence, it is the interplay between the two effects, the decreasing of $1/\sqrt{N}$ and the concomitant increasing of $E_{r_1}$ (both of which are brought about by the synchronization of the response to the periodic bias) that leads to the minimum in the error $E_r$. This effect is therefore closely related to that of stochastic resonance [10] which is also related to the synchronization of the response to the periodic field. However, it should be stressed that the difference here is that in our system the signal to be detected is a dc field—not the *known* periodic bias.

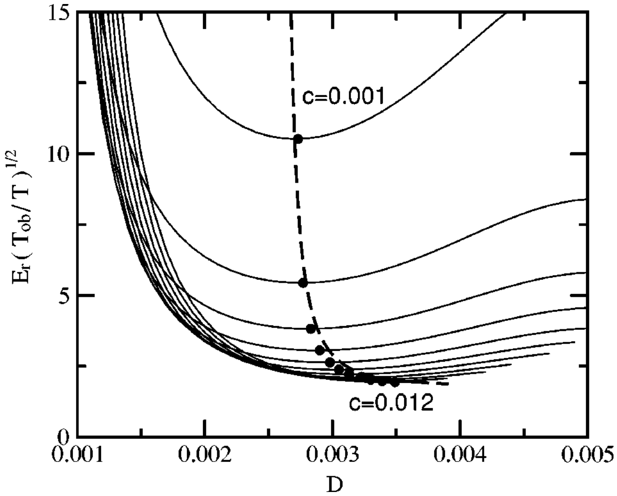


FIG. 3. The dimensionless standard deviation (or the error) $E_r$ vs noise intensity parameter $D$, computed via Eq. (9). $A=0.35$, $\Omega=0.0025$, and $c=0.001, \ldots, 0.012$ with the step 0.001 (from top to bottom). The dashed curve is the locus of critical $D$ values from Eq. (12) and the dots are the actual critical values.

The fact that the minimum occurs as a result of synchronization enables us to develop a criterion for predicting the position of the minimum. For synchronization to occur the system must be capable of making a transition between the states every half cycle. For a symmetric system ($c=0$) the criterion for this to occur is that the maximum Kramers rate $W_{max}$ attained during a complete cycle of the periodic force must be greater than (or of the same order) as the frequency of the external field $\Omega$. More specifically, the criterion for the *onset* of synchronization can be stated as [11] $\sqrt{2\pi} W_{max}\delta t \approx 1$ where $\delta t \propto \Omega^{-1}$. For the asymmetric case under study here, a similar criterion can be found but now we require that $W_{max} = \min(W_{12max}, W_{21max})$, where $W_{12max}$ and $W_{21max}$ are the maximum Kramers rates (over one cycle) for transitions from state 1 and state 2, respectively. Using this criterion the noise intensities that lead to the minimum are those that satisfy the equality

$$\sqrt{2\pi} W_{12max}\delta t_1 = 1, \tag{12}$$

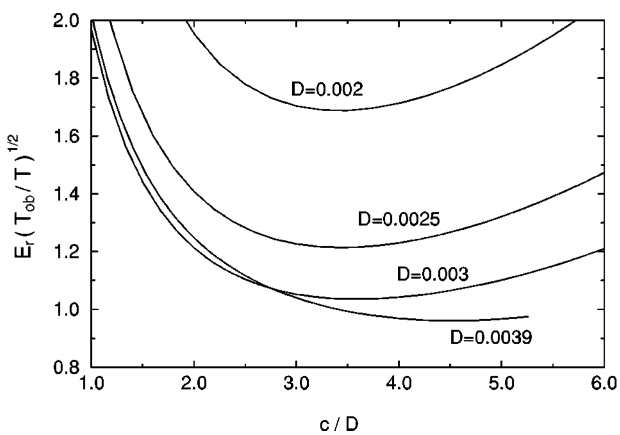


FIG. 4. The dimensionless standard deviation (or the error) $E_r$ vs $c/D$, computed via Eq. (9). $A=0.35$, $\Omega=0.0025$, $a=1$, and $b=1$.

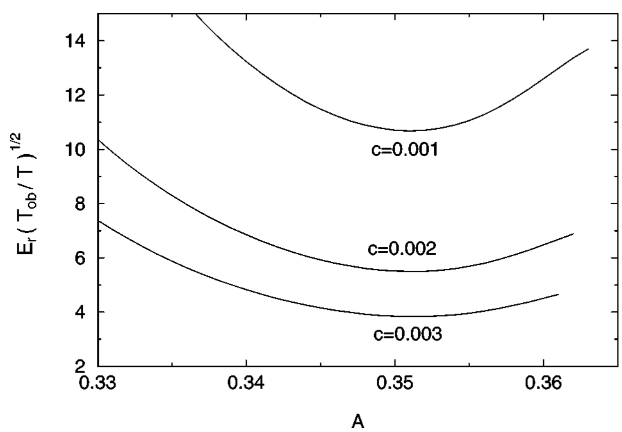


FIG. 5. The dimensionless standard deviation (or the error) $E_r$ vs $A$, computed via Eq. (9). $\Omega=0.0025$, and $D=0.003$.

where $\delta t=\Omega^{-1}\sqrt{D/A\Delta}$, $\Delta$ being the separation (in the $x$ coordinate) between the potential maximum and minimum when the Kramers rate is maximized. This criterion is tested in Fig. 3. The dashed line shows the values of $D$ obtained from Eq. (12) for various $c$ and the solid dots show the actual values. Clearly, a reasonably good agreement is obtained. This indicates that, just like stochastic resonance, the minimum in the measurement error occurs when a stochastic time scale (the inverse Kramers rate) is matched to a deterministic time scale (external bias period).

Of course, one could reasonably expect a minimum in the error for fixed noise, if the bias and target signal parameters are varied. Figures 3, 4, and 5 show that this is indeed the case.

The results of this paper are critical to the operation of nonlinear detectors as threshold crossing devices, with the RTDs (and the associated mean values) taken to be the quantities that quantify small perturbations. Since this technique is quite easy to implement experimentally [4], it is of interest to understand the optimal regime of bias, given fixed detector parameters. In particular, the reference bias amplitude should be subthreshold with $A/D \gg 1$ (the *strongly nonlinear* regime). In this regime, the dependence of $\Delta T$ on $c$ is almost exponential [8], corresponding to an extremely sensitive detection regime for very weak asymmetries (dc signals). However, because the signals are subthreshold, the noise floor becomes critical to the dynamics. If the noise is too weak, hopping events are infrequent and this leads to significant errors in computing $\Delta T$. We have seen that increasing $D$ increases the number of threshold crossing events, leading to a regime of synchronization and hence a reduction in the measurement error. We therefore conclude that, under the appropriate conditions, the measurement error can be reduced by the addition of noise.

Finally we note that, given the above discussion, the effect will not occur for strongly suprathreshold bias signals (the case considered in our earlier work [4]). In this regime, the system is always completely synchronized (in fact, one obtains a Gaussian distribution of residence times) and adding noise cannot increase the number of transitions in a fixed observation period $T_{ob}$. Hence, the only effect of the noise is to increase the error.

We gratefully acknowledge support from the Office of Naval Research, Code 331.

[1] P. Reimann, Phys. Rep. **361**, 57 (2002).
[2] P. Jung and R. Bartussek, in *Flucuations and Order: The New Synthesis*, edited by M. Millonas (Springer, New York, 1988); A.R. Bulsara, M.E. Inchiosa, and L. Gammaitoni, Phys. Rev. Lett. **77**, 2162 (1996); M. Inchiosa, A.R. Bulsara, and L. Gammaitoni, Phys. Rev. E **55**, 4049 (1997).
[3] L. Gammaitoni and A.R. Bulsara, Phys. Rev. Lett. **88**, 230601 (2002).
[4] A. R. Bulsara, C. Seberino, L. Gammaitoni, M. F. Karlsson, B. Lundqvist, and J. W. C. Robinson, Phys. Rev. E **67**, 016120 (2003).
[5] M. Inchiosa and A.R. Bulsara, Phys. Rev. E **58**, 115 (1998).
[6] W. G. Geyger, *Nonlinear Magnetic Control Devices* (McGraw-Hill, New York, 1964); P. Ripka, Sensors and Actuators A **33**, 129 (1992).
[7] See, e.g., P. Jung, Phys. Rep. **234**, 175 (1993).
[8] A. P. Nikitin, N. G. Stocks, and A. R. Bulsara, Phys. Rev. E **68**, 016103 (2003).
[9] H. Risken, *The Fokker-Planck Equation* (Springer-Verlag, Berlin, 1984).
[10] For a good review, see L. Gammaitoni, P. Hanggi, P. Jung, and F. Marchesoni, Rev. Mod. Phys. **70**, 223 (1998).
[11] N.G. Stocks, Il Nuovo Cimento **17D**, 925 (1995).